Definable envelopes in groups having a simple theory

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Abstract. Let G be a group having a simple theory. For any nilpotent subgroup N of class n, there is a definable nilpotent subgroup E of G which is virtually 'nilpotent of class at most $2n$' and finitely many translates of which cover N. The group E is definable using parameters in N, and normalised by $N_G(N)$. If S is a soluble subgroup of G of derived length ℓ, there is a definable soluble subgroup F which is virtually 'soluble of derived length at most 2ℓ' and contains S. The group F is definable using parameters in S and normalised by $N_G(S)$. Analogous results are shown in the more general setting where the ambient group G is defined by the conjunction of infinitely many formulas in a structure having a simple theory. In that case, the envelopes E and F are defined by the conjunction of infinitely many formulas.

1. Introduction

When studying a group, a model theorist focuses on sets that are definable by formulas. It happens that in a group G, one finds a subgroup H of particular interest, having a given property P such as abelian, nilpotent, soluble etc. One then tries to find a definable group which also has property P and contains H. We call any group containing H an *envelope* of H. Finding a definable envelope of H with property P is possible when the ambient group is well behaved:

A group has the property MC if it satisfies the minimality condition on centralisers, that is if every strictly decreasing chain of centralisers $C_G(A_1) \supset C_G(A_2) \supset \cdots$ has a finite length. An abelian subgroup of an MC group is contained in an abelian definable subgroup (the centre of its centraliser). Stable groups are particular examples of MC groups. Poizat showed that if G is stable, then every nilpotent subgroup of G is contained in a nilpotent definable subgroup of the same nilpotency class, and every soluble subgroup of G is contained in a soluble definable subgroup of the same derived length (see [Poi87] ; the results also appear in [Wag97]). In a recent paper, Altınel and Baginski have shown that a nilpotent subgroup of an MC group is enveloped by a nilpotent definable subgroup of the same nilpotency class (see [AB]).

Wider than the class of stable groups is the class of groups that do not have the independence property. In the case where G is a group without the independence property, Shelah [She09] has shown that if there is an infinite abelian subgroup of G, then there is one which is definable. This was improved by Aldama [Ald] who showed that any nilpotent subgroup of G of nilpotency class n is enveloped by a definable nilpotent group of class n. In those two cases, the parameters needed to define the enveloping group may lie in a saturated extension of the ambient group, *i.e.* the envelope is of the form $G \cap \mathbf{E}$, where $\mathbf{E}$ is a definable subgroup of a saturated extension $\mathbf{G}$ of G.

Another important class of groups extending the class of stable groups is the class of groups having a simple theory, which includes in particular all pseudofinite simple groups (*i.e.* pseudofinite groups without non-trivial normal subgroups, see [Wil] and [Hru] ; we shall keep those two wordings to avoid any confusion between simple groups and groups having a simple theory). The previous results however do not hold in general if G has merely a simple theory. For instance they do not hold if G is an infinite extra-special p-group for some odd prime p, *i.e.* if every g in G has order p and in addition the centre of G is cyclic of order p and equals G'. Such a group has a simple theory

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(actually its theory is supersimple of SU-rank 1, see [MS] and Appendix), is nilpotent of class 2, and possesses an infinite abelian subgroup; but any abelian definable subgroup of G is finite by [Plo].

However, if G is a group having a supersimple theory of finite rank which eliminates the quantifier $\exists^\infty$, it is shown in [EJMR] that a soluble subgroup S of G of derived length ℓ is contained in a definable soluble subgroup E such that the derived series of S and E share the same number of infinite factors. The authors of [EJMR] derive that the soluble radical of G is definable and soluble. If G merely has a simple theory, it is shown in [Mil12] that an abelian subgroup of G is always contained in a definable subgroup H of G that has a normal finite subgroup N such that H/N is abelian (we call such a group H *finite-by-abelian*).

Let us recall that a group is FC if each of its elements has finitely many conjugates. Finite-by-abelian groups are FC-groups. When we were looking for the narrowest possible definable group which envelopes an abelian, a nilpotent, or a soluble subgroup of a group having a simple theory, it turned out that the problem was conceptually simpler in a more general setting involving FC, FC-nilpotent, and FC-soluble groups instead of abelian, nilpotent, and soluble groups. Our main results are the following two theorems which hold for any group G having a simple theory:

Theorem 1.1 (definable envelope around a nilpotent or FC-nilpotent subgroup)**.**

(1) *Let A be an FC-nilpotent subgroup of G of class n. There is a definable (using parameters in A) FC-nilpotent subgroup of G of class n which is normalised by $N_G(A)$ and contains A.*
(2) *Let N be a nilpotent subgroup of G of nilpotency class n. There is a definable (using parameters in N) nilpotent subgroup F of G, which is virtually $2n$-nilpotent, and finitely many translates of which cover N. Moreover, F is normalised by any element of $N_G(N)$.*

Theorem 1.2 (definable envelope around a soluble or FC-soluble subgroup)**.**

(1) *Let B be an FC-soluble subgroup of G of length n. There is a definable (using parameters in B) FC-soluble subgroup of G of length ℓ which is normalised by $N_G(B)$ and contains B.*
(2) *Let S be a soluble subgroup of G of derived length ℓ. There is a definable (using parameters in S) soluble subgroup R of G, which is virtually 2ℓ-soluble, and contains S. Moreover, R is normalised by any element of $N_G(S)$.*

From Theorem 1.1, one can derive a positive answer to a question raised by Elwes, Jaligot, Ryten and Macpherson in [EJMR]:

Corollary 1.3. *The Fitting subgroup of a supersimple group of finite rank is nilpotent and definable.*

The following result follows from Theorem 1.2:

Corollary 1.4. *The soluble radical of a supersimple group of arbitrary rank is soluble and definable.*

The proofs of Corollary 1.3 and 1.4 can be found in [Mil13]. In the last section, we extend the previous results to the case where the ambient group G is merely given by a bounded intersection of formulas in a simple theory:

Theorem 1.5. *Let N be a nilpotent subgroup of G of class n. There is a type-definable (with parameters in N) nilpotent subgroup F which is virtually $2n$-nilpotent, and a finite number of translates of which cover N. The group F is normalised by $N_G(N)$.*

Theorem 1.6. *Let S be a soluble subgroup of G of derived length ℓ. There is a type-definable (with parameters in S) soluble subgroup R, which is virtually 2ℓ-soluble, and contains S. The group R is normalised by $N_G(S)$.*

2. Preliminaries on FC-nilpotency and FC-solubility

Let G be a group and $g, h \in G$. We call $g^h = h^{-1}gh$ the *h-conjugate* of g and $C_G(h) = \{g \in G : h^g = h\}$ the *centraliser of h in G*. If N is a normal subgroup of G, we denote by hN the *coset* $\{hn : n \in N\}$ and by $C_G\big(\{hN\}\big)$ the set $\{g \in G : h^gN = hN\}$. Let H be a subgroup of G, we write H^g for the set $\{h^g : h \in H\}$. We call the set $N_G(H) = \{g \in G : H^g = H\}$ the *normaliser of H in*

G. The G-conjugacy class of h is the set $h^G = \{h^g : g \in G\}$. We call G an *FC-group* if for every $g \in G$, the conjugacy class g^G is finite, or equivalently if the index $[G : C_G(g)]$ is finite.

Following Haimo in [Hai], we define the *FC-centraliser of a subgroup H of G* by:

$$FC_G(H) = \big\{g \in G : H/C_H(g) \text{ is finite}\big\}.$$

If N is a normal subgroup of H, we extend this definition by putting

$$FC_G(H/N) = \big\{g \in G : H\big/C_H\big(\{gN\}\big) \text{ is finite}\big\}.$$

The *FC-centre of G* is defined by:

$$FC(G) = FC_G(G).$$

Then, we define the *nth FC-centre of G* by the following induction on n:

$$FC_0(G) = \{1\} \text{ and } FC_{n+1}(G) = FC_G\big(G/FC_n(G)\big).$$

Finally, the *FC-normaliser of H in G* is defined by:

$$FN_G(H) = \big\{g \in G : H^g/H \cap H^g \text{ and } H/H \cap H^g \text{ are finite}\big\}.$$

These are all subgroups of G. The chain $FC_1(G) \leqslant \cdots \leqslant FC_n(G)$ is an ascending chain of characteristic subgroups of G (in particular, $FC_n(G)$ is normal in G and its inductive definition makes sense).

Lemma 2.1. *Let G be a group, $g \in G$ an element, $N \leqslant H \leqslant G$ subgroups and $n \in \omega$. Then,*

1. *$N_G(H)$ normalises $FC_G(H)$,*
2. *If N and H are normal in G, then $C_G(\{gN\})$ is a subgroup of $C_G\big(\{gH\}\big)$,*
3. *$H \cap FC_n(G)$ is a subgroup of $FC_n(H)$,*
4. *If N is a finite normal subgroup of H, then $FC_G(H/N) = FC_G(H)$.*

Proof. (1) Let $h \in N_G(H)$. As $C_H(g^h) = C_H(g)^h$, if $H/C_H(g)$ is finite, then so is $H/C_H(g^h)$. It follows that h normalises $FC_G(H)$.

(2) If h centralises $\{gN\}$, then $[g, h]$ is an element of N, hence of H, so h centralises $\{gH\}$.

(3) If h is an element of $H \cap FC(G)$, then $G/C_G(h)$ is finite. As $H/C_H(h)$ embeds in $G/C_G(h)$, the element h belongs to $FC(H)$ as well. Now suppose that $FC_n(G) \cap H$ is a subgroup of $FC_n(H)$ and let h belong to $FC_{n+1}(G) \cap H$. It follows that $G\big/C_G\big(\{hFC_n(G)\}\big)$ is finite, so $H\big/C_H\big(\{hFC_n(G)\}\big)$ is finite. As h is an element of H, one has $C_H\big(\{hFC_n(G)\}\big) = C_H\big(\{h(FC_n(G) \cap H)\}\big)$. By induction hypothesis and point 2, the group $H\big/C_H\big(\{hFC_n(H)\}\big)$ must be finite, which proves that h belongs to $FC_{n+1}(H)$.

(4) For any $g \in G$, the group $C_H(g)$ is a subgroup of $C_H(\{gN\})$, so if $H/C_H(g)$ is finite, then so is $H/C_H(\{gN\})$. It follows that the group $FC_G(H)$ is a subgroup of $FC_G(H/N)$. If x is an element of the latter, there is a subgroup F of finite index in H which centralises the finite set $\{xN\}$. So x^F is finite, and so is x^H. □

Definition 2.2 (Haimo [Hai])**.** A group is called *FC-nilpotent* if one of the following equivalent facts holds:

1. There is an FC-central series of finite length, *i.e.* a sequence of normal subgroups of G
$$\{1\} = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_n = G$$
such that H_{i+1}/H_i is included in the FC-centre of G/H_i for every i in $\{0, \ldots, n-1\}$.
2. The sequence of iterated FC-centres ends on G after n steps. We call the least such n the *FC-class of G*, or simply its *class* when there is no ambiguity.

Proof. The sequence of iterated FC-centres is an FC-central series, so (ii) implies (i). Reciprocally, if (i) holds, an induction on n shows that $H_i \leqslant FC_i(G)$ holds for all $i \leqslant n$. This shows that $FC_n(G)$ equals G. □

Definition 2.3 (adapted from Duguid, McLain [DM])**.** A group G is called *FC-soluble* if there exists a normal *FC-series* of finite length, *i.e.* a finite sequence of normal subgroups $G_0, G_1, \dots, G_\ell$ of G such that

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_\ell = \{1\}$$

and such that G_i/G_{i+1} is an FC-group for all i. We call the least such natural number ℓ the *FC-length of* G, or simply its *length*.

Remark 2.4. For a group G, the requirements of having a finite series with abelian factors, and a finite series with abelian factors whose members are in addition normal in G are equivalent. This may not be true in the case of an FC-series. We have modified the original definition in [DM] and we do require here that the FC-series of G consist of subgroups which are normal in G.

Finite groups and abelian groups are FC-groups. FC-nilpotent groups of class 1 and FC-soluble groups of length 1 coincide with FC-groups. Nilpotent groups of class n are FC-nilpotent of class at most n. A soluble group G of derived length ℓ is FC-soluble of length at most ℓ. In fact, if G has a derived series with $d \leqslant \ell$ infinite factors, then G is FC-soluble of length d. FC-nilpotent groups of class n are FC-soluble of length at most n. We recall from [Neu, Theorem 3.1]:

Theorem 2.5 (Neumann)**.** *Let G be an FC-group whose conjugacy classes are finite and bounded. Then the derived subgroup G' is finite.*

The direct image of an FC-group by a group homomorphism is an FC-group, and the pre-image of an FC-group by a group homomorphism having a finite kernel is FC. As a corollary:

Lemma 2.6. *Let G and H be two groups and $f : G \to H$ an homomorphism.*

(1) *If G is FC-nilpotent, so is $f(G)$. If the kernel of f is finite and $f(G)$ is FC-nilpotent, so is G.*
(2) *If G is FC-soluble, so is $f(G)$. If the kernel of f is finite and $f(G)$ is FC-soluble, so is G.*

In particular, subgroups and quotient groups of FC-nilpotent groups are FC-nilpotent, and subgroups and quotient groups of FC-soluble groups are FC-soluble.

The centre of a nilpotent group is non-trivial. For an FC-nilpotent group:

Lemma 2.7. *Let G be any group.*

(1) *If G is FC-nilpotent and infinite, then $FC(G)$ is infinite.*
(2) *$FC_n(G)$ is FC-nilpotent of class at most n.*

Proof. If $FC(G)$ is finite, then $FC_2(G)$ equals $FC(G)$ by Lemma 2.1(4), so G is finite.

For point (ii), one has $FC_n(G) \subset FC_n\big(FC_n(G)\big)$ by Lemma 2.1(3) and the other inclusion is obvious □

3. Groups having a simple theory

Simple theories are pointed out in [She80] as a wider class than, but still analoguous to the class of stable theories, which in turn were introduced in [She69] as a generalisation of Morley's totally transcendental theories (see [Mor]). In Morley's own word, the terminology transcendental is suggested by the theory of algebraically closed field. In Shelah's word, the rank of a stable theory is a generalisation of Morley's rank of transcendence. Let us recall that one way to define the dimension of an algebraic variety V over an algebraically closed field is inductively, saying that V has dimension at least $n+1$ if there are infinitely many disjoint sub-varieties of V having dimension at least n. Note that these sub-varieties are defined by specifying a parameter in the polynomial equations defining V, hence are uniformly defined by a single formula. This is the point of view that we chose here to define a simple theory.

We recall in this section all the elementary definitions and known results on groups having a simple theories that will be needed in the paper and refer to [She80], [Pil98] and [Wag05, Wag00] for more details. In a given language, let $\psi(\bar{x})$ and $\phi(\bar{x}, \bar{y})$ be first order formulas, T a complete theory and M a model of T. Let k be a natural number. We define the *$D_{\phi,k}$-rank* of ψ with respect to T by induction:

(1) $D(\psi,\phi,k) \geqslant 0$ if ψ is consistent with T;
(2) $D(\psi,\phi,k) \geqslant n+1$ if there is a sequence $\bar{a}_0, \bar{a}_1, \dots$ such that $D\big(\psi(\bar{x}) \wedge \phi(\bar{x},\bar{a}_i), \phi, k\big)$ is at least n for all i and if the formulas $\phi(\bar{x},\bar{a}_0), \phi(\bar{x},\bar{a}_1), \dots$ are k-inconsistent with T.

If X is a set defined by a formula $\psi(\bar{x})$, we write $D(X,\phi,k)$ for $D(\psi,\phi,k)$. In other words, the set X has $D_{\phi,k}$ rank at least $n+1$ if one can find an infinite family of k-inconsistent subsets of X which are all uniformly definable using the single formula ϕ, and whose $D_{\phi,k}$ rank is at least n. Note that the rank $D(\psi,\phi,k)$ takes three arguments. A basic induction shows that it is increasing in the first argument (with respect to logical implication), decreasing in the second one, and increasing in the third one (with respect to the natural order on $\mathbf{N}$). We say that a subset of the cartesian product M^n is *type-definable* if it is defined by the conjunction of a bounded infinity of first order formulas. The following lemma is a slight generalisation of [Wag00, Remark 4.1.5], with a parameter $\bar{b}$ allowed in the second argument of the rank D.

Lemma 3.1 (Wagner). *Let $\ell, m \in \omega$, let $\bar{y}$ be an ℓ-tuple of variables and $\bar{z}$ an m-tuple of variables. Let $\phi_{\bar{z}}(\bar{x},\bar{y})$ be a formula. For any formula $\psi(\bar{x},\bar{y})$ and $k \in \omega$, the set $\big\{(\bar{a},\bar{b}) \in M^{\ell+m} : D(\psi(\bar{x},\bar{a}), \phi_{\bar{b}}, k) \geqslant n\big\}$ is type-definable.*

Proof. By induction on n. For $n = 0$, the set $\big\{(\bar{a},\bar{b}) \in M^{\ell+m} : D(\psi(\bar{x},\bar{a}), \phi_{\bar{b}}, k) \geqslant 0\big\}$ is defined by the type $\{\exists\bar{x}\psi(\bar{x},\bar{a})\} \cup T$. Let us assume that $\big\{(\bar{a},\bar{b},\bar{c}) \in M^{2\ell+m} : D(\psi(\bar{x},\bar{a}) \wedge \phi_{\bar{b}}(\bar{x},\bar{c}), \phi_{\bar{b}}, k) \geqslant n\big\}$ is defined by the type $\pi(\bar{a},\bar{b},\bar{c})$. The condition '$\phi_b(\bar{x},\bar{a}_0), \phi_{\bar{b}}(\bar{x},\bar{a}_1), \dots$ are k-inconsistent with T', which we write $I(\bar{a}_0,\bar{a}_1,\dots,\bar{b})$, is expressed by the conjunction

$$\bigwedge_{\chi\in T} \chi \wedge \bigwedge_{0\leqslant i_1<i_2<\cdots<i_k} \neg(\exists\bar{x})\big(\phi_{\bar{b}}(\bar{x},\bar{a}_{i_1}) \wedge \cdots \wedge \phi_{\bar{b}}(x,\bar{a}_{i_k})\big).$$

It follows that the condition $D(\psi(\bar{x},\bar{a}), \phi_{\bar{b}}, k) \geqslant n+1$ is equivalent to

$$(\exists\bar{a}_0\exists\bar{a}_1\exists\bar{a}_2\cdots) I(\bar{a}_0,\bar{a}_1,\bar{a}_2,\dots,\bar{b}) \wedge \bigwedge_{i\geqslant 0} \pi(\bar{a},\bar{b},\bar{a}_i),$$

which can be shown to be a type-definable condition, by a compactness argument. □

Definition 3.2 (Shelah [She80]). A complete theory is called *simple* if for every formula ϕ and every natural number k, the $D_{\phi,k}$-rank of all of its formulas is a natural number. A structure is called *simple* if its first order theory is so.

Lemma 3.3 (Shelah [She80]). *Let X and Y be two definable subsets of some given structure. The rank $D(X \cup Y,\phi,k)$ equals the maximum of $D(X,\phi,k)$ and $D(Y,\phi,k)$.*

Proof. By a basic induction on the rank $D(X \cup Y,\phi,k)$. □

Let $\phi(\bar{x},\bar{y})$ be a formula and g a function symbol. We write $g^{-1}\phi(\bar{x},\bar{y})$ for the formula $\phi(g(\bar{x}),\bar{y})$. If the language contains the language of groups, we write $z\phi(\bar{x},\bar{y})$ or simply $z\phi$ for the formula $\phi(z^{-1}\bar{x},\bar{y})$, where z is thought of as a new parameter variable of arity 1. Likewise, we often talk of $D_{z\phi,2}$-rank, for example. We take the opportunity to stress the following crucial lemma, which we could not find anywhere in the literature:

Lemma 3.4. *In any structure, let X and Y be two definable sets and let g be a definable map from X to Y.*

(1) *If g is surjective, then $D(X,g^{-1}\phi,k) \geqslant D(Y,\phi,k)$.*
(2) *If g has fibres of size at most n, then $D(X,g^{-1}\phi,k) \leqslant D(Y,\phi,kn)$.*

Proof. (1) Assume that g is surjective. We proceed by induction on the rank $D(Y,\phi,k)$. If Y is consistent, then so is X. If $D(Y,\phi,k) \geqslant n+1$ holds, then there are formulas $\phi(\bar{x},\bar{a}_1), \phi(\bar{x},\bar{a}_2), \dots$ which are k-inconsistent, with each $\phi(Y,\bar{a}_i)$ having $D_{\phi,k}$-rank at least n. By induction hypothesis, their preimages $g^{-1}\phi(\bar{x},\bar{a}_1), g^{-1}\phi(\bar{x},\bar{a}_2), \dots$ witness that the $D_{g^{-1}\phi,k}$-rank of X is at least $n+1$.

For point (2), there is no harm in assuming that g is onto. If X is consistent, then so is Y. We go on inductively and suppose that $D(X, g^{-1}\phi, k)$ is at least $m+1$. This provides us with k-inconsistent sets $X_0, X_1, \dots$ defined by $g^{-1}\phi(\bar{x}, \bar{a}_0), g^{-1}\phi(\bar{x}, \bar{a}_1), \dots$ with the rank $D(X \cap X_i, g^{-1}\phi, k)$ being at least m. By induction hypothesis, as g is onto, $D\big(Y \cap g(X_i), \phi, kn\big) \geqslant m$ holds. Let I be a subset of $\mathbf{N}$ of cardinality kn and suppose that there is some element $\bar{y}$ of $\bigcap_{i\in I} g(X_i)$. Let $\bar{x}_1, \dots, \bar{x}_p$ be the list of all pre-images of $\bar{y}$ by g. Every x_i belongs to finitely many distinct sets among $(X_i)_{i\in I}$, let us say say n_i distinct ones, so we have $n_1 + \cdots + n_p \geqslant |I|$. As p is at most n, at least one n_i must be greater or equal than k, which contradicts that the sets $X_0, X_1, \dots$ are k-inconsistent. This shows that the sets $g(X_0), g(X_1), \dots$ are kn-inconsistent and that the rank $D(Y, \phi, kn)$ is at least $m+1$. □

It is worth mentioning that one could weaken the definability assumption on g in the previous lemma and require only that the images and pre-images by g of uniformly definable sets be uniformly definable. In particular, one could take g to be an automorphism of the structure. As our proofs on groups heavily rely on passing to quotient groups, we recall that simplicity is preserved under taking a quotient by a definable equivalence relation.

Corollary 3.5 (Shelah). *Let M be a simple structure and let E be a definable equivalence relation on M. Then the disjoint union of M and M/E is a simple structure. (The language considered is the language on M extended by a predicate for M/E and a function symbol for the canonical surjection from M to M/E).*

Proof. As E is definable in M, the new language does not induce new formulas on M, so M is simple in the extended language. By Lemma 3.3, it is enough to show that M/E is simple. If $\phi(\bar{x}, \bar{y})$ is any formula, we write $\phi_E(\bar{x}, \bar{y})$ for the formula $(\exists \bar{z})\big(\bar{x}E\bar{z} \wedge \phi(\bar{z}, \bar{y})\big)$. By Lemma 3.4(1) applied to the canonical surjection from M to M/E, the rank $D(M, \phi, k)$ is at least $D(M/E, \phi_E, k)$, so M/E is simple. □

Lemma 3.6 (adapted from Pillay [Pil98, Remark 3.12]). *In any group, let G be a definable subgroup, and let H be a definable subgroup of G relatively defined in G by the formula ϕ (*i.e.* H is the subset of G whose elements verify ϕ). Then the coset space G/H is finite if and only if H and G have the same $D_{z\phi,2}$-rank.*

Proof. For all formulas $\psi(\bar{x}, \bar{y})$ and all natural numbers k, the cosets of H have the same $D_{z\psi,k}$-rank by Lemma 3.4. If G/H is finite, then G is covered by finitely many cosets of H, so G and H have the same $D_{z\psi,k}$-rank by Lemma 3.3. Reciprocally, if G/H is infinite, then G is covered by infinitely many pairwise disjoint cosets of H. This witnesses that the rank $D(G, z\phi, 2)$ is at least $D(H, z\phi, 2) + 1$. □

The dimension of an algebraic variety V is also defined by the Krull dimension of the algebra of the polynomials over V, which provides a bound on the length of a descending chain of sub-varieties of V. In a group having a simple theory, the intersections of subgroups which are uniformly defined by a fixed formula satisfy a descending chain condition 'up to finite index':

Theorem 3.7 (Wagner [Wag00, Theorem 4.2.12]). *Let $\phi(x, \bar{y})$ be a fixed formula, let G be a group having a simple theory and let $H_1, H_2, \dots$ be a family of subgroups of G defined respectively by formulas $\phi(x, \bar{a}_1), \phi(x, \bar{a}_2) \dots$. If $G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots$ is a descending chain of finite intersections of H_i, then there exists a natural number n such that G_m has a finite index in G_n for all $m \geqslant n$.*

Proof. By adding new subgroups in the chain $G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots$, we may assume without loss of generality that for every $i \geqslant 1$, there exists a parameter $\bar{b}_i$ and a subgroup K_i of G defined by the formula $\phi(x, \bar{b}_i)$, such that G_{i+1} equals $G_i \cap K_i$. As $D(G_1, z\phi, 2)$ is finite, there is a natural number n such that G_m and G_n have the same $D_{z\phi,2}$-rank for all $m \geqslant n$. By Lemma 3.6, the quotient G_n/G_m is finite for every $m \geqslant n$. □

Two subgroups of a given group G are called *commensurable* if the index of their intersection is finite in each of them. Commensurability is an equivalence relation on the set of subgroups of G. A family of subgroups of G is called *uniformly commensurable* if its members are pairwise commensurable and if the correspondent family of finite indexes is bounded by a natural number. The following result appears in its original form in [Sch]; the generalised version stated bellow is the one given by Peter Neumann in [Ne], with some additional details coming from [Wag00, Theorem 4.2.4] and its proof ($\langle\mathfrak{H}\rangle$ in [Wag00, Theorem 4.2.4] is replaced by $\mathfrak{H}^4$ here, which stands for the set $\bigcup_{H_i\in\mathfrak{H}} H_1H_2H_3H_4$. This is immediate from the construction of N in Wagner's proof).

Theorem 3.8 (Neumann's version of Schlichting)**.** *Let G be a group and let $\mathfrak{H}$ be a family of uniformly commensurable subgroups of G. Then there exists a subgroup N of G which is commensurable with the members of $\mathfrak{H}$ and invariant under the action of the automorphisms of the structure G (possibly extending the group structure) which stabilise the family $\mathfrak{H}$ setwise. The inclusions $\bigcap\mathfrak{H} \subset N \subset \mathfrak{H}^4$ hold. Moreover, N is a finite extension of a finite intersection of subgroups belonging to $\mathfrak{H}$. In particular, if $\mathfrak{H}$ consists of definable groups, then N is also definable.*

4. Definable envelopes

Theorem 4.1. *In a group G having a simple theory, let H be a definable subgroup (using parameters in a finite set A). The FC-centraliser, FC-normaliser and iterated FC-centres of H are definable subgroups of G (using parameters in the set A). In particular, $FC(H)$ has finite and bounded conjugacy classes, and the set $\{H^g : g \in FN_G(H)\}$ consists of uniformly commensurable subgroups.*

Proof. Let $\mathbf{G}$ be an $\aleph_0$-saturated *elementary extension* of G. That is, if $\varphi(x_1,\dots,x_n)$ is a formula without parameters and $g_1,\dots,g_n$ belong to G, then $\varphi(g_1,\dots,g_n)$ holds in G if and only if it holds in $\mathbf{G}$.

We shall prove that $FC_G(H)$ is definable. Let $\phi(x,g)$ be a formula defining the group $C_G(g)$, let $\psi(x)$ be a formula defining H and $\mathbf{H}$ the set of realisations of $\psi(x)$ in $\mathbf{G}$. We denote the rank $D(\mathbf{H}, z\phi(x,g), 2)$ by n. By Lemma 3.6, we have

$$FC_{\mathbf{G}}(\mathbf{H}) = \Big\{g \in \mathbf{G} : D\big(C_{\mathbf{H}}(g), z\phi(x,g), 2\big) = n\Big\} = \Big\{g \in \mathbf{G} : D\big(C_{\mathbf{H}}(g), z\phi(x,g), 2\big) \geqslant n\Big\}$$

As $\mathbf{G}$ is $\aleph_0$-saturated, one can identify the type-definable sets with parameters in A with the consistent types over A. On the one hand, by Lemma 3.1, $FC_{\mathbf{G}}(\mathbf{H})$ is a type-definable subgroup of $\mathbf{G}$, so $FC_{\mathbf{G}}(\mathbf{H})$ is closed in the topology generated by formulas over A so is a compact set of types. On the other hand, the condition $\Big|\mathbf{H}/C_{\mathbf{H}}(g)\Big| = n$ is an A-definable condition on g, so $FC_{\mathbf{G}}(\mathbf{H})$ is covered by the countable union $\bigcup_{n\in\mathbf{N}^*}\Big\{g \in G : \Big|\mathbf{H}/C_{\mathbf{H}}(g)\Big| = n\Big\}$ of clopen sets. By the Compactness Theorem, finitely many of these clopen sets must cover $FC_{\mathbf{G}}(\mathbf{H})$. Thus, $FC_{\mathbf{G}}(\mathbf{H})$ is defined by a single formula $\varphi(x)$ using parameters in A. We claim that

$$FC_G(H) = \{g \in G : \varphi(g) \text{ holds in } G\}.$$

If g is an element of G such that $\varphi(g)$ holds, then g belongs to $FC_{\mathbf{G}}(\mathbf{H})$ so $g^{\mathbf{H}}$ is finite, and g^H is also finite. Reciprocally, if g is an element of G such that g^H is finite, it equals some natural number, say m. The condition $\Big|g^H\Big| = m$ is expressible by a formula using parameters in G. As $\mathbf{G}$ is an elementary extension of G, this formula also holds in $\mathbf{G}$, so that $\Big|g^{\mathbf{H}}\Big| = m$. It follows that g belongs to $FC_{\mathbf{G}}(\mathbf{H})$, and $\varphi(g)$ holds in $\mathbf{G}$, hence in G.

We have just shown that $FC_G(H)$ is definable. By Corollary 3.5, if N is a definable normal subgroup of H, the structure $G \cup H/N$ is also simple. A similar argument shows that $FC_G(H/N)$ is definable. By an immediate induction, the iterated FC-centres of H are also definable.

A similar argument works for $FN_G(H)$ as well: Let $\phi(x,g)$ be a formula defining the group H^g, let $\psi(x)$ be a formula defining H and $\mathbf{H}$ the set of realisations of $\psi(x)$ in $\mathbf{G}$. We denote the rank $D(\mathbf{H}, z\phi(x,g), 2)$ by n. By Lemma 3.6, we have

$$FN_{\mathbf{G}}(\mathbf{H}) = \Big\{g \in \mathbf{G} : D\big(\mathbf{H}^g \cap \mathbf{H}, z\phi(x,g), 2\big) \geqslant n \text{ and } D\big(\mathbf{H}^{-g} \cap \mathbf{H}, z\phi(x,g), 2\big) \geqslant n\Big\}$$

On the other hand, $FN_{\mathbf{G}}(\mathbf{H})$ is covered by the following countable union of clopen sets:

$$\bigcup_{n,m\in\mathbf{N}^*}\Big\{g\in G:\Big|\mathbf{H}/\mathbf{H}^g\cap\mathbf{H}\Big|=n\text{ and }\Big|\mathbf{H}^g/\mathbf{H}^g\cap\mathbf{H}\Big|=m\Big\}.$$

□

Theorem 4.2. *Let G be a group having a simple theory and let N be an FC-nilpotent subgroup of class n. There is an FC-nilpotent definable subgroup E of G of class n which envelopes N. Moreover, E is normalised by any element of $N_G(N)$.*

Proof. By Theorem 4.1, the FC-centralisers, FC-normalisers and iterated FC-centres of definable subgroups of G are definable. We build recursively a decreasing chain of definable subgroups $B_1\geqslant\cdots\geqslant B_n$ such that for every i in $\{1,\dots,n\}$,

(1) B_i envelopes N,
(2) $N_G(N)$ normalises B_i.
(3) $FC_i(B_i)$ envelopes $FC_i(N)$.

We begin by building the group B_1. Let $a_1,\dots,a_p$ be elements of $FC(N)$ such that $C_G(a_1,\dots,a_p,a)$ has a finite index in $C_G(a_1,\dots,a_p)$ for any a in $FC(N)$. They do exist by Theorem 3.7. We denote by C_1 the subgroup $C_G(a_1,\dots,a_p)$, which is minimal 'up to finite index'. Note that $N/C_N(a_i)$ is finite for every a_i, so $N/N\cap C_1$ is also finite. For every element g of $N_G(N)$, the subgroup C_1^g equals $C_G(a_1^g,\dots,a_p^g)$. As $N_G(N)$ normalises $FC(N)$, the subgroups C_1^g and C_1 are commensurable by minimality of C_1. It follows that the FC-normaliser of C_1 contains $N_G(N)$. As $FN_G(C_1)$ is definable by Theorem 4.1, the family of $N_G(N)$-conjugates of C_1 must be uniformly commensurable. This allows us to apply Theorem 3.8 to the set $\big\{C_1^g:g\in N_G(N)\big\}$ and to find some definable subgroup D_1 which is commensurable with C_1 and which in addition is normalised by $N_G(N)$. As $N/N\cap C_1$ is finite, $N/N\cap D_1$ is finite too, so the subgroup $N\cdot D_1$ is a finite union of cosets of D_1, hence is definable. We define B_1 to be $N\cdot D_1$ and claim that its FC-centre contains $FC(N)$: if g belongs to $B_1\setminus FC(B_1)$, then $B_1/B_1\cap C_G(g)$ is infinite. But B_1 and C_1 are commensurable, so $C_1/C_1\cap C_G(g)$ is infinite too, and g cannot belong to $FC(N)$ by minimality of C_1. This completes the first step.

Now we assume that $B_1,\dots,B_{k-1}$ are built, and we build B_k. The subgroup B_{k-1} normalises $FC_{k-1}(B_{k-1})$ by Lemma 2.1(1), so $B_{k-1}/FC_{k-1}(B_{k-1})$ is a group. Note that the disjoint union of G and $B_{k-1}/FC_{k-1}(B_{k-1})$ is a simple structure by Corollary 3.5, so we may apply the descending chain condition of Theorem 3.7 in G, taking parameters in $B_{k-1}/FC_{k-1}(B_{k-1})$. For any element b of B_{k-1}, we write $\bar b$ for the class of b in the quotient group $B_{k-1}/FC_{k-1}(B_{k-1})$. Let $b_1,\dots,b_m$ be elements of $FC_k(N)$ such that $C_{B_{k-1}}(\bar b_1,\dots,\bar b_m,\bar b)$ has a finite index in $C_{B_{k-1}}(\bar b_1,\dots,\bar b_m)$ for any b in $FC_k(N)$. Let us denote by C_k the group $C_{B_{k-1}}(\bar b_1,\dots,\bar b_m)$. The quotient $N\big/C_N\big(\big\{b_iFC_{k-1}(N)\big\}\big)$ is finite for every b_i, and $FC_{k-1}(B_{k-1})$ envelopes $FC_{k-1}(N)$ by induction hypothesis, so $N\big/C_N\big(\big\{b_iFC_{k-1}(B_{k-1})\big\}\big)$ is also finite by Lemma 2.1(2). It follows that $N/C_N(\bar b_1,\dots,\bar b_m)$ is finite. So is $N/N\cap C_k$. We can once again show that the groups in $\big\{C_k^g:g\in N_G(N)\big\}$ are uniformly commensurable, apply Theorem 3.8 to the set of $N_G(N)$-conjugates of C_k, and find some definable subgroup D_k normalised by $N_G(N)$ and commensurable with C_k. We denote by B_k the definable subgroup $N\cdot D_k$, which is a subgroup of B_{k-1}. To finish the proof, we just need to show that $FC_k(B_k)$ envelopes $FC_k(N)$. If g is an element of $B_k\setminus FC_k(B_k)$, then $B_k\big/B_k\cap C_G\big(\big\{gFC_{k-1}(B_k)\big\}\big)$ is infinite. As B_k is a subgroup of B_{k-1}, Lemma 2.1(3) yields

$$FC_{k-1}(B_{k-1})\cap B_k\leqslant FC_{k-1}(B_k),$$

so the group $B_k\big/B_k\cap C_G\big(\big\{g(FC_{k-1}(B_{k-1})\cap B_k)\big\}\big)$ is infinite. As g belongs to B_k, we have

$$B_k\cap C_G\Big(\Big\{g(FC_{k-1}(B_{k-1})\cap B_k)\Big\}\Big)=B_k\cap C_G\Big(\Big\{gFC_{k-1}(B_{k-1})\Big\}\Big)=B_k\cap C_G(\bar g).$$

It follows that $B_k/B_k\cap C_G(\bar g)$ is infinite. As B_k and C_k are commensurable, the quotient $C_k/C_k\cap C_G(\bar g)$ is infinite too, so g is not an element of $FC_k(N)$ by minimality of C_k. This completes the recursive construction of the groups B_i.

It follows that $FC_n(B_n)$ is a definable FC-nilpotent subgroup of class at most n that contains $FC_n(N)$, hence N. By Lemma 2.1(i), the group $FC_n(B_n)$ is normalised by $N_G(B_n)$ hence by $N_G(N)$. $\square$

Remark 4.3. It may be desirable to have a closer look at the parameters necessary to define the definable envelope of N in the previous proof. The only places where we may have used parameters outside N is when we applied Schlichting's Theorem, the first time being to the set $\big\{C_1^g : g \in N_G(N)\big\}$, where C_1 is definable using parameters in N. This provides a finite extension D_1 of a group D definable using parameters in $N_G(N)$ (D is the intersection of finitely many $N_G(N)$-conjugates of C_1). But the automorphisms of the structure G fixing N pointwise stabilise the set $\big\{C_1^g : g \in N_G(N)\big\}$ setwise (we shall say that it is N-*invariant*). By Schlichting's Theorem again, the group D_1 is N-invariant, hence definable using parameters in N. It follows that every definable group considered in the proof is in fact definable using parameters in N.

Remark 4.4. When applying Schlichting's Theorem, instead of considering the action of $N_G(N)$ by conjugation, we can modify the construction of E and consider the action of the automorphisms of the structure G which leave N invariant. This ensures that the definable envelope E is σ-invariant whenever σ is an automorphism of G such that N is σ-invariant.

In [Wag00], Wagner defines a notion of 'almost' centraliser, which suggests a notion of 'almost-nilpotent' group suitable for groups which are hyperdefinable in a simple theory. He shows that the 0-hyperdefinable connected component of a 0-hyperdefinable almost-nilpotent group of class n is nilpotent of class at most $2n$ (we recall that the 0-hyperdefinable connected component is by definition the intersection of all the subgroups having bounded index which are hyper-definable with no parameters).

If G is a definable group, we call the intersection of all the subgroups of G having a finite index in G that are definable using parameters in A the *connected component of G over A*. We denote it by G^0_A, or simply G^0 when A is the empty set.

Proposition 4.5. *In a group having a simple theory, if H is an FC-nilpotent subgroup of class n that is definable using parameters in a set A, then H^0_A is nilpotent of nilpotency class at most $2n$.*

Proof. We may add the elements of A in the language and assume that A is empty. If K and L are two subsets of H, we call *the generating set of* $[K, L]$ the set of all commutators $k^{-1}\ell^{-1}k\ell$ when k ranges over K and ℓ over L. We denote by $[K, L]$ the subgroup that these commutators generate. We define $[K,_n L]$ recursively by putting

$$[K,_n L] = [K, L] \text{ for } n = 1, \text{ and } [K,_{n+1} L] = \big[[K,_n L], L\big].$$

Note that the generating set of $[FC(H), FC(H]$ is finite by Theorem 2.5 and Theorem 4.1. Following exactly the proof of [Wag00, Proposition 4.4.10.3] while remarking that 'bounded' in Wagner gets replaced by 'finite', we claim that the generating set of $[FC(H), H^0]$ is also finite. We do not give a proof here, as it involves technicalities that we have not defined. Because $[FC(H), H^0]$ is normal in H, the centraliser in H of $[FC(H), H^0]$ has a finite index in H. It follows that $C_H\big(\big[FC(H), H^0\big]\big)$ contains H^0, so that the group $[FC(H), H^0, H^0]$ is trivial. We show inductively on n that $[FC_n(H),_{2n} H^0]$ is trivial. Let us assume that $[FC_k(H),_{2k} H^0]$ is trivial. By the first step of the induction, the generating set of $\big[FC\big(H/FC_k(H)\big), \big(H/FC_k(H)\big)^0\big]$ is finite, so the generating set of $\big[FC_{k+1}(H)/FC_k(H), H^0/FC_k(H)\big]$ is also finite. It follows that the centraliser of $\big[FC_{k+1}(H), H^0\big]FC_k(H)\big/FC_k(H)$ has a finite index in H, hence contains H^0. Thus we have

$$\big[FC_{k+1}(H), H^0, H^0\big] \leqslant FC_k(H).$$

Applying the induction hypothesis, we get

$$\big[FC_{k+1}(H),_{2k+2} H^0\big] \leqslant \big[FC_k(H),_{2k} H^0\big] = \{1\}.$$

This shows that $[FC_n(H),_{2n} H^0]$ is trivial. As $FC_n(H)$ equals H, the group $[H^0,_{2n} H^0]$ is trivial. $\square$

Let P be a group property. We call a group G *virtually* P if it has a definable subgroup of finite index with property P. We shall say that G is n*-nilpotent* if it is nilpotent of nilpotency class at most n, and ℓ*-soluble* if it is soluble of derived length at most ℓ.

Corollary 4.6. *In a group G having a simple theory, let N be a nilpotent subgroup of nilpotency class n. There is a definable (using parameters in N) nilpotent subgroup F of G, which is virtually $2n$-nilpotent, and finitely many translates of which cover N. Moreover, F is normalised by any element of $N_G(N)$.*

Proof. Let $\mathbf{G}$ be a $|N|$-saturated elementary extension of G. By Theorem 4.2, there is an FC-nilpotent subgroup of $\mathbf{G}$ of class n which contains N, is normalised by $N_G(N)$. Let us call it $\mathbf{H}$. By Remark 4.3, $\mathbf{H}$ is definable with parameters in N. By Proposition 4.5, the subgroup $\mathbf{H}_N^0$ is nilpotent of nilpotency class at most $2n$. This is witnessed by the formula saying that the commutator of every $2n+1$ elements is trivial. As $\mathbf{H}_N^0$ is the intersection of groups definable using parameters in N, by the Compactness Theorem and the saturation assumption, it follows that there is a subgroup of finite index in $\mathbf{H}$ that is $2n$-nilpotent, and definable using parameters in N. It follows that $\mathbf{H}$ has a definable $2n$-nilpotent normal subgroup $\mathbf{E}$ of finite index. Let $\mathbf{F}$ denote the *Fitting subgroup* of $\mathbf{H}$, that is, the subgroup of $\mathbf{H}$ generated by all the normal nilpotent subgroups of $\mathbf{H}$. The group $\mathbf{F}$ is a characteristic nilpotent subgroup of $\mathbf{H}$, hence is also normalised by $N_{\mathbf{G}}(N)$. As $\mathbf{F}$ is a finite union of translates of $\mathbf{E}$, it is definable. For any subset $\mathbf{X}$ of $\mathbf{G}$ defined in $\mathbf{G}$ by a formula $\phi(x)$, we denote by X the set $\{g \in G : \phi(g) \text{ hold}\}$. As G is an elementary substructure of $\mathbf{G}$, the group F has the same first order properties as $\mathbf{F}$. In particular, F has a finite index in H. As H contains N, finitely many translates of F cover N. □

Corollary 4.7. *In a group having a simple theory, a nilpotent subgroup N of nilpotency class n is contained in a $3n$-soluble virtually $2n$-nilpotent definable subgroup, using parameters in N.*

Proof. By Corollary 4.6, there is a definable $2n$-nilpotent subgroup F such that the quotient $N/N \cap F$ is finite. The N*-core of* F defined by $F_N = \bigcap_{n\in N} F^n$ is thus a finite intersection of definable groups, hence definable, and $2n$-nilpotent as a subgroup of F. As N normalises F_N, the product $N \cdot F_N$ is soluble of derived length at most $n+2n$. As a union of finitely many cosets of F_N, the group $N \cdot F_N$ is definable. □

Corollary 4.8. *In a group G having a simple theory, let N be a nilpotent normal subgroup of nilpotency class n. Then, there is a normal $3n$-nilpotent subgroup of G enveloping N and definable using parameters in N.*

Proof. As above, we consider the definable group $F_N \cdot N$. As N and F_N are both normal subgroups in $F_N \cdot N$, by Fitting's Lemma, $F_N \cdot N$ is nilpotent of class at most $n + 2n$. We take the G-core $(F_N \cdot N)_G$. □

Corollary 4.9. *Let G be a group having a simple theory. If G has an infinite nilpotent subgroup, then G has an infinite finite-by-abelian definable subgroup.*

Proof. By Corollary 4.6, the group G has an infinite definable subgroup N which is nilpotent, and hence FC-nilpotent. The FC-centre of N is definable by Theorem 4.1, hence finite-by-abelian by Theorem 2.5, and infinite by Lemma 2.7. □

We go on with the soluble subgroups of a group G having a simple theory. It was shown laboriously in [Mil12, Corollary 5.12] that a soluble subgroup of G whose derived length is ℓ is enveloped by a definable subgroup H that has a finite series $H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_{2\ell-1} = \{1\}$ of length $2\ell - 1$ consisting of definable subgroups whose factors H_i/H_{i+1} are finite-by-abelian. We provide a better version here.

Theorem 4.10. *Let G be a group having a simple theory and let S be an FC-soluble subgroup of length ℓ. Then S is enveloped by a definable (using parameters in S) FC-soluble subgroup E of length ℓ, the members of whose FC-series are normal definable subgroups (using parameters in S). Moreover, E is normalised by any element of $N_G(S)$.*

Proof. Let $S = S_0 \trianglerighteq S_1 \trianglerighteq \cdots \trianglerighteq S_\ell = \{1\}$ be an FC-series for S. We recall that every S_i is normal in S. We set Z_0 equal to the trivial subgroup and recursively build an ascending chain of definable subgroups $Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots \trianglelefteq Z_\ell$ such that for every i in $\{1, \dots, \ell\}$,

(1) Z_i is normal in Z_ℓ,
(2) Z_i contains $S_{\ell-i}$,
(3) $N_G(S_{\ell-i})$ normalises Z_i,
(4) Z_i/Z_{i-1} is an FC-group.

Let us first build Z_1. By Theorem 3.7, let $a_1, \dots, a_p$ be elements of $S_{\ell-1}$ such that $C_G(a_1, \dots, a_p, a)$ has a finite index in $C_G(a_1, \dots, a_p)$ for all a in $S_{\ell-1}$. We write C_1 for the centraliser $C_G(a_1, \dots, a_p)$. As $S_{\ell-1}$ is an FC-group, the quotient $S_{\ell-1}/S_{\ell-1} \cap C_1$ is finite. Let g be an element of $N_G(S_{\ell-1})$. As g normalises $S_{\ell-1}$, the subgroup C_1^g is the centraliser of elements of $S_{\ell-1}$, hence is commensurable with C_1 by minimality of C_1. As previously seen, the set $\left\{C_1^g : g \in N_G(S_{\ell-1})\right\}$ consists of uniformly commensurable subgroups. By Theorem 3.8, there is a definable subgroup B_1 which is commensurable with C_1 and normalised by $N_G(S_{\ell-1})$. The subgroup $B_1 \cdot S_{\ell-1}$ is a finite extension of B_1, hence definable, and commensurable with B_1. Let us call it D_1. Let Z_1 be its FC-centre and let $N_1 = N_G(Z_1)$. We show that Z_1 contains $S_{\ell-1}$: if g belongs to $D_1 \setminus FC(D_1)$, then $D_1/D_1 \cap C_G(g)$ is infinite and so is $C_1/C_1 \cap C_G(g)$. By minimality of C_1, the element g is not in $S_{\ell-1}$.

Now we assume that $Z_1, \dots, Z_k$ are built, and we put $N_i = N_G(Z_i)$ for each i. Note that $N_1 \cap \cdots \cap N_k$ contains S. For any b in N_k, we write $\bar{b}$ for the class of b in the quotient group N_k/Z_k. Let $b_1, \dots, b_p$ be elements of $S_{\ell-k-1}$ such that $N_1 \cap \cdots \cap N_k \cap C_G(\bar{b}_1, \dots, \bar{b}_p)$ is minimal up to finite index. Let us denote by C_{k+1} the group $N_1 \cap \cdots \cap N_k \cap C_G(\bar{b}_1, \dots, \bar{b}_p)$. By induction hypothesis, the group Z_k contains $S_{\ell-k}$, so $Z_k S_{\ell-k-1}/Z_k$ is an FC-group. In particular, $\left(N_k \cap C_G(\bar{b}_i)\right)/Z_k \cap Z_k S_{\ell-k-1}/Z_k$ has a finite index in $Z_k S_{\ell-k-1}/Z_k$ for every i in $\{1, \dots, p\}$. Note that $N_1 \cap \cdots \cap N_k$ contains S and Z_k, and hence $Z_k S_{\ell-k-1}$. It follows that $(Z_k S_{\ell-k-1}/Z_k) \cap (C_{k+1}/Z_k)$ has a finite index in $Z_k S_{\ell-k-1}/Z_k$. If g normalises $S_{\ell-k-1}$, then C_{k+1}^g and C_{k+1} are commensurable by minimality of C_{k+1}; note that C_{k+1}^g is a subgroup of $N_1 \cap \cdots \cap N_k$. As previously seen, the set $\left\{C_{k+1}^g : g \in N_G(S_{\ell-k-1})\right\}$ consists of uniformly commensurable subgroups, so we apply Theorem 3.8 to $\left\{C_{k+1}^g : g \in N_G(S_{\ell-k-1})\right\}$ inside $N_1 \cap \cdots \cap N_k$ and find a definable subgroup B_{k+1} of $N_1 \cap \cdots \cap N_k$ which is normalised by $N_G(S_{\ell-k-1})$ and commensurable with C_{k+1}. Let D_{k+1} be the group $B_{k+1} S_{\ell-k-1}/Z_k$. It is a finite extension of the definable group B_{k+1}/Z_k, hence definable, and commensurable with C_{k+1}/Z_k. Let Z_{k+1} be the preimage in G of the FC-centre of D_{k+1}. We claim that Z_{k+1} contains $S_{\ell-k-1}$: if not, then there is some element g of $S_{\ell-k-1} \setminus Z_{k+1}$, thus $(B_{k+1} S_{\ell-k-1})/(B_{k+1} S_{\ell-k-1}) \cap C_G(\bar{g})$ is infinite, so $C_{k+1}/C_{k+1} \cap C_G(\bar{g})$ is also infinite, hence g cannot belong to $S_{\ell-k-1}$. This completes the induction. □

Corollary 4.11. *In a group having a simple theory, a definable FC-soluble subgroup H has an FC-series whose members are normal definable subgroups of H.*

Proof. We apply Theorem 4.10 to H inside H. □

Note that Proposition 4.5 shows in particular that an FC-nilpotent group having a simple theory has a definable nilpotent subgroup of finite index. For an FC-soluble group, we have the following:

Corollary 4.12. *In a group G having a simple theory, let H be a definable (using parameters in the set A) FC-soluble subgroup of length ℓ. Then H has a definable (using parameters in A) subgroup S of finite index which is 2ℓ-soluble.*

Proof. Let $H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_\ell = \{1\}$ be an FC-series for H with every H_i being normal in H. By Corollary 4.11, we may assume that the subgroups H_i are definable. As each quotient H_i/H_{i+1} is FC, it follows from Theorem 4.1 that the centralisers of every element $\bar{g}$ of H_i/H_{i+1} have a finite and bounded index in H_i/H_{i+1}. By Theorem 2.5, the quotient group $[H_i, H_i]H_{i+1}/H_{i+1}$ is finite. As H normalises the finite set $[H_i, H_i]H_{i+1}/H_{i+1}$, the centraliser of $[H_i, H_i]H_{i+1}/H_{i+1}$ has a finite index in H. As it is definable using parameters in A, it must contain H_A^0. Thus we have

$$\left[[H_i, H_i], H_A^0\right] \leqslant H_{i+1}.$$

For any subgroup K of G, let us write $K^{(0)} = K$ and $K^{(k+1)} = [K^{(k)}, K^{(k)}]$. With i equal to 0, the above shows that $(H_A^0)^{(2)}$ is a subgroup of H_1. Let us show inductively in $n \leq \ell$ that $(H_A^0)^{(2k)}$ is a subgroup of H_k. Assume that this is done until the step n. Then we have

$$\left(H_A^0\right)^{(2n+2)} = \Big[\big[H_A^0)^{(2n)}, (H_A^0)^{(2n)}\big] , \big[H_A^0)^{(2n)}, (H_A^0)^{(2n)}\big] \Big] \leqslant \big[[H_n, H_n], H_A^0\big] \leqslant H_{n+1}.$$

This shows that the derived subgroup $(H_A^0)^{(2\ell)}$ is trivial. Because solubility of derived length 2ℓ is expressible by a first order formula, going to an $\aleph_0$-saturated extension of H, using the Compactness Theorem and coming back to H, we find a subgroup X of finite index in H which is soluble of derived length at most 2ℓ and definable using parameters in A. □

Corollary 4.13. *In a group G having a simple theory, let S be a soluble subgroup of derived length ℓ. There is a definable (using parameters in S) soluble subgroup E, which is virtually 2ℓ-soluble, and contains S. Moreover, E is normalised by any element of $N_G(S)$.*

Proof. By Theorem 4.10 and Corollary 4.11, there is a definable FC-soluble subgroup H of derived length ℓ which contains S, has a definable FC-series, and is normalised by $N_G(S)$. By Corollary 4.12, the group H has a definable subgroup of finite index which is 2ℓ-soluble, hence H has a normal such subgroup, which we call F. Let R be the *soluble radical of* H, which is generated by all the normal soluble subgroups of H. It is a soluble, and a characteristic subgroup of H. In particular, it is normalised by $N_G(H)$, hence by $N_G(S)$. As S and R are both soluble and because S normalises R, the product $S \cdot R$ is soluble. As R contains F, the subgroup $S \cdot R$ is the union of finitely many cosets of F, hence definable, and virtually 2ℓ-soluble. □

Corollary 4.14. *In a group G having a simple theory, let S be a normal soluble subgroup of derived length ℓ. There is a definable (using parameters in S) normal subgroup enveloping S which is 3ℓ-soluble and virtually 2ℓ-soluble.*

Proof. Corollary 4.13 provides us with a definable normal virtually 2ℓ-soluble subgroup E of G which contains S. So there is a definable soluble subgroup R of derived length at most 2ℓ such that $S/S\cap R$ is finite. It follows that $S \cdot R$ is the union of finitely many cosets of R, hence a definable subgroup. Because R normalises S, the product $S \cdot R$ is also soluble of derived length at most $\ell + 2\ell$. □

5. Type-definable envelopes

In a stable structure with no group law given *a priori*, under a certain geometric configuration, one can build a non-trivial group G (see [Pil96, chapter 5]). G is not necessarily defined by one formula but rather by infinitely many. In the case of a simple theory T, an analoguous result is established in [BTW] (see also [PKM] for a different approach). It is an open question whether, in T, a group H that is the conjunction of infinitely many definable sets is the conjunction of definable groups. Should that be true, Theorem 4.2 and Theorem 4.10 would immediately yield that a nilpotent subgroup of H is contained in a type-definable virtually nilpotent subgroup, and a soluble subgroup of H is contained in a type-definable soluble one. These are the results that we establish in this section.

Throughout the section, we fix an infinite non-countable cardinal κ, a simple theory T, and a κ-saturated model M of T. A set is called *bounded* if its cardinality is strictly less than κ. We call an *n-type* any bounded set $\pi(x_1, \ldots, x_n)$ of consistent formulas in n variables $x_1, \ldots, x_n$. As M is κ-saturated, we may identify any partial n-type over a bounded set of parameters with the set of its realisations in M^n. We say that G is a *type-definable group in* M if there is

(1) a 1-type $\pi(x)$, such that G is the subset of M whose elements satisfy every formula in π, and
(2) a definable subset D of M containing G, and a definable composition law $\times$ from $D \times D$ to M such that $(G, \times)$ is a group.

As noticed in [Poi87], assuming the group law $\times$ and the set D to be type-definable instead of definable is equivalent by a compactness argument.

For the development of a suitable version of Theorems 4.2 and 4.10 for type-definable groups, we introduce new definitions. Let G be any group. We call G a *BC group* if for every g in G, the conjugacy class g^G is bounded, or equivalently if the centraliser $C_G(g)$ has a bounded index in G. These BC groups were first considered by Tomkinson [Tom], who calls them *κC groups*, and then independently by Wagner [Wag00], who calls them *almost nilpotent groups of class one*. Let H be a subgroup of G. Following [Wag00, p. 119] with adapted notation and terminology, we define the *BC-centraliser of H in G* by:

$$BC_G(H) = \big\{g \in G : H/C_H(g) \text{ is bounded}\big\}.$$

If N is a normal subgroup of H, we extend this definition by putting

$$BC_G(H/N) = \big\{g \in G : H\big/C_H\big(\{gN\}\big) \text{ is bounded}\big\}.$$

The *BC-centre of* G is defined by:

$$BC(G) = BC_G(G).$$

Then, we define the *nth BC-centre of* G by the following induction on n:

$$BC_0(G) = \{1\} \text{ and } BC_{n+1}(G) = BC_G\big(G/BC_n(G)\big).$$

Finally, the *BC-normaliser of H in G* is defined by:

$$BN_G(H) = \big\{g \in G : H^g/H \cap H^g \text{ and } H/H \cap H^g \text{ are bounded}\big\}.$$

If G is a type-definable group in M and H a relatively definable subgroup of G, then G/H is bounded if and only if it is finite by the Compactness Theorem and saturation hypothesis. As $C_K(g)$ is relatively definable in K for any type definable subgroup K of G, it follows that $BC_G(K) = FC_G(K)$. It is show in [Wag00, Proposition 4.4.10] that for a type-definable subgroup K of G, the groups $BC_G(K)$, $BN_G(K)$ and $BC_n(K)$ are type-definable. For a relatively definable subgroup of G, with a proof similar to Theorem 4.1 we get:

Theorem 5.1. *Let G be a type-definable group in M and H a relatively definable subgroup of G (using parameters in the set A). Then one has $BC(H) = FC(H)$, $BN_G(H) = FN_G(H)$ and $BC_n(H) = FC_n(H)$ and all those subgroups are relatively definable in G (using parameters in A).*

A group is called *hyperdefinable in M* if it is a quotient group of the form G/H where G is a type-definable group in M and H a type-definable normal subgroup of G. Neumann's Theorem 2.5 for FC-groups has an analogue for BC-groups, at least in a group having a simple theory:

Lemma 5.2. *Let G be a hyperdefinable group in M. If G is a BC group, then its derived subgroup G' is bounded.*

Proof. We denote by G^{00} the intersection of all the hyperdefinable subgroups of G that have a bounded index in G using types without parameters. By [Wag00, Proposition 4.4.10.3], the group $\big[BC(G), G^{00}\big]$ is bounded. As G is a BC group, $BC(G)$ equals G, so $[G, G^{00}]$ is bounded. We denote the latter by S. Let I be a transversal for G^{00} in G. Let g and h be in G, and let $g = kx$ and $h = \ell y$ where k and ℓ belong to I and x and y belong to G^{00}. Applying commutator identities, we have

$$[g, h] = [kx, \ell y] = [kx, y] \cdot [kx, \ell]^y = [kx, y] \cdot \big([k, \ell]^x \cdot [x, \ell]\big)^y.$$

This yields

$$\begin{aligned}[g, h] &= [kx, y] \cdot (xy)^{-1} \cdot [k, \ell] \cdot xy \cdot [x, \ell]^y \\ &= [kx, y] \cdot \big[xy, [k, \ell]^{-1}\big] \cdot [k, \ell] \cdot [x, \ell]^y\end{aligned}$$

and

$$[g, h] \in S^2 \cdot I^4 \cdot S.$$

But G^{00} has a bounded index in G by [Wag00, Proposition 4.4.5], so I is bounded. As S is also bounded, the generating set of $[G, G]$ is bounded. □

Definition 5.3 (from Wagner [Wag00])**.** A group is called *BC-nilpotent* if one of the following equivalent facts holds:

(1) There is a BC-central series of finite length, *i.e.* a sequence of normal subgroups of G

$$\{1\} = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_n = G$$

such that H_{i+1}/H_i is included in the BC-centre of G/H_i for every i in $\{0, \dots, n-1\}$.

(2) The sequence of iterated BC-centres ends on G after n steps. We call the least such n the *BC class of G*, or simply its *class* when there is no ambiguity.

Definition 5.4. A group G is called *BC-soluble* if there exists a normal *BC-series* of finite length, *i.e.* a finite sequence of normal subgroups $G_0, G_1, \dots, G_\ell$ of G such that

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_\ell = \{1\}$$

and such that G_i/G_{i+1} is a BC group for all i. We call the least such natural number ℓ the *BC length of G*, or simply its *length*.

We cite the version of Theorem 3.7 suitable for type-definable groups.

Theorem 5.5 (Descending chain condition 'up to finite index' adapted from [Wag00, Theorem 4.2.12])**.** *Let $\varphi(x, y)$ be a formula and let G be a type-definable group in M. Let $H_1, H_2, \dots$ be a family of subgroups of G defined respectively by $\varphi(G, a_1), \varphi(G, a_2) \dots$. If $G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots$ is a descending chain of subgroups of G such that every G_i is the intersection of finitely many H_j, then there exists a natural number n such that G_m has a finite index in G_n for all $m \geqslant n$.*

The proofs of the following two theorems follow those of Theorems 4.2 and 4.10 using Theorem 5.1, and Theorem 5.5 instead of Theorem 4.1 and Theorem 3.7.

Theorem 5.6. *Let G be a type-definable group in M and let N be an FC-nilpotent subgroup of class n. Then N is enveloped by a relatively definable (with parameters in N) FC-nilpotent subgroup F of class n. Moreover, F is normalised by any element of $N_G(N)$.*

Theorem 5.7. *Let G be a type-definable group in M and let S be an FC-soluble subgroup of length ℓ. Then S is enveloped by a relatively definable (with parameters in S) FC-soluble subgroup R of length ℓ, the members of whose FC series are relatively definable subgroups. Moreover, R is normalised by any element of $N_G(S)$.*

Proposition 5.8. *Let G be a type-definable group in M. If G is BC-nilpotent of class n, then G has a relatively definable subgroup of finite index which is $2n$-nilpotent.*

Proof. We denote by G^{00} the intersection of all the type-definable subgroups of H using no parameters and which have a bounded index in G. By [Wag00, Proposition 4.4.10.3], the subgroup G^{00} is nilpotent of nilpotency class at most $2n$. This is witnessed by a partial $(2n+1)$-type saying that the commutator of every $2n+1$ elements of G^{00} is trivial. By the Compactness Theorem, there is a definable set X containing G such that the group law is defined over the product of any 2^{2n+1} elements of X (who may lie outside X though). By the Compactness Theorem again, there is a definable subset Y of X containing G^{00}, the commutator of every $2n+1$ elements of which is trivial. By definition of G^{00}, the set Y contains a type-definable subgroup of G that has a bounded index in G. It follows that a finite number of translates of Y by elements $g_1, \dots, g_m$ of G cover G. This implies that the group $\langle Y \cap G\rangle$ equals $(h_1 Y \cup \cdots \cup h_p Y) \cap G$ for some finite subset $\{h_1, \dots, h_p\}$ of $\{g_1, \dots, g_m\}$. Thus, $\langle Y \cap G\rangle$ is relatively definable in G, has a finite index in G, and is $2n$-nilpotent. □

Corollary 5.9. *Let G be a type-definable group in M and let N be a nilpotent subgroup of class n. There is a type-definable (with parameters in N) nilpotent subgroup F of G which is virtually $2n$-nilpotent, and a finite number of translates of which cover N. The group F is normalised by $N_G(N)$.*

Proof. Same proof as for Corollary 4.6. Note that the transfer argument in Corollary 4.6 is not needed here as M is already assumed to be sufficiently saturated. □

Corollary 5.10. *Let G be a type-definable group in M and let N be a normal nilpotent subgroup of nilpotency class n. There is a type-definable (with parameters in N) normal $3n$-nilpotent subgroup enveloping N.*

Proof. Corollary 5.9 provides a type-definable nilpotent subgroup F of G such that the quotient $N/N \cap F$ is finite. It follows that the core $\bigcap_{g \in N} F^g$ is a finite intersection of conjugates of H, hence type-definable, $2n$-nilpotent (as a subgroup of F), and normalised by N. Let us call it F_N. By Fitting's Lemma, the group $N \cdot F_N$ is nilpotent of class at most $n + 2n$. Note that $N/N \cap F_N$ is finite, so $N \cdot F_N$ is a finite extension of F_N, hence type-definable. The group $(N \cdot F_N)_G$ is as desired. □

We end with tackling the soluble subgroups. From Theorem 5.7, we deduce:

Corollary 5.11. *Let G be a type-definable group in M. If G is FC-soluble, then it has an FC-series whose members are relatively definable normal subgroups of G.*

Proposition 5.12. *Let H be a type-definable group in M and G a type-definable BC-soluble subgroup of length ℓ. Then G has a type-definable normal subgroup S of finite index which is 2ℓ-soluble.*

Proof. Let $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_\ell = \{1\}$ be a BC series for G such that every G_i is normal in G. By Corollary 5.11, we may assume that the subgroups G_i are type-definable. Without loss of generality, we may also assume that the types defining each G_i use no parameters (or add a bounded number of parameters in the language). As each quotient G_i/G_{i+1} is BC, the centralisers of every element $\bar{h}$ of G_i/G_{i+1} have a bounded index in G_i/G_{i+1}. By Lemma 5.2, the quotient group $[G_i, G_i]G_{i+1}/G_{i+1}$ is bounded. As G normalises the group $[G_i, G_i]G_{i+1}/G_{i+1}$, the centraliser of $[G_i, G_i]G_{i+1}/G_{i+1}$ has a bounded index in G. As it is definable without parameters, it must contain G^{00}. Thus, for all $i \leqslant n - 1$, we have

$$\big[[G_i, G_i], G^{00}\big] \leqslant G_{i+1}.$$

With i equal to 0, this shows that $(G^{00})^{(2)}$ is a subgroup of G_1. Let us show inductively on $n \leqslant \ell$ that $(G^{00})^{(2n)}$ is a subgroup of G_n. Assume that this is done until the step n. Then we have

$$\left(G^{00}\right)^{(2n+2)} = \Big[\, \big[G^{00})^{(2n)}, (G^{00})^{(2n)}\big] \,, \big[G^{00})^{(2n)}, (G^{00})^{(2n)}\big] \,\Big] \leqslant \big[[G_n, G_n], G^{00}\big] \leqslant G_{n+1}.$$

This shows that the derived subgroup $(G^{00})^{(2\ell)}$ is trivial. By a compactness argument, as in the nilpotent case, we find a type-definable subgroup E of G of finite index which is soluble of derived length no greater than 2ℓ. The G-core of E meets our requirement. □

Corollary 5.13. *Let G be a type-definable group in M and let S be a soluble subgroup of derived length ℓ. There is a type-definable (with parameters in S) soluble subgroup R, which is virtually 2ℓ-soluble, and contains S. The group R is normalised by $N_G(S)$.*

Corollary 5.14. *Let G be a type-definable group in M and let S be a normal soluble subgroup of derived length ℓ. There is a type-definable (with parameters in S) normal subgroup of G which envelopes S, is 3ℓ-soluble, and virtually 2ℓ-soluble.*

Appendix. On infinite extra-special p-groups

A referee pointed out that the theory of an infinite extra-special p-group being supersimple of SU-rank 1 follows indeed from Proposition 3.11 and Lemma 4.1 of [MS]. We provide an elementary alternative proof below, which ensures that the upper bound provided in Proposition 4.5, Corollary 4.6 and Corollary 4.13 are in some sense optimal.

Definition 5.15 (Hall, Higman [HH])**.** For an odd prime p, a group G is called an *infinite extra-special p-group* if G is infinite, $g^p = 1$ for every g in G, the centre of G is cyclic of order p and equals G'.

Note that theses axioms are expressible in first order logic. This axiomatises a complete theory according to [Fel]. We denote by V an infinite vector space over a finite field F equipped with a non-degenerate skew-symmetric bilinear form [,].

Lemma 5.16. *Let $(\lambda_1, \dots, \lambda_n)$ be in F^n. If $a_1, \dots, a_n$ are linearly independent in V, then the solution set of the equations $[x, a_1] = \lambda_1, \dots, [x, a_n] = \lambda_n$ is an affine subspace of V whose underlying vector space has codimension n in V.*

Proof. As the bilinear form is non-degenerate, the linear forms mapping x to $[x, a_i]$ are linearly independent. □

Lemma 5.17. *The theory of V in the language of F-vector-spaces together with a binary function symbol $[\ ,\ ]$ (where F is identified with a given subspace of V) is complete and eliminates quantifiers.*

Proof. We may without loss of generality add the elements of F to the language. If T is the theory of V, let A and B be two models of T. Let $a_1, \dots, a_n$ be elements of A, let $b_1, \dots, b_n$ be elements of B such that there exist a local isomorphism σ between $a_1, \dots, a_n$ and $b_1, \dots, b_n$; that is, σ is a bijection between $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ such that for all atomic formula $\varphi(x_1, \dots, x_n)$ in n variables, $\varphi(a_1, \dots, a_n)$ holds in A if and only if $\varphi(b_1, \dots, b_n)$ holds in B. If a_{n+1} belongs to A, we show that σ can be extended to a local isomorphism having domain $\{a_1, \dots, a_{n+1}\}$. This is enough by [Poi85, Fraïssé, théorème 2.02]. We may replace $a_1, \dots, a_n$ by a basis of $\mathrm{Span}(a_1, \dots, a_n)$ and assume without loss of generality that $a_1, \dots, a_n$ are linearly independent. It follows that $b_1, \dots, b_n$ also are linearly independent. There exist scalars $\lambda_1, \dots, \lambda_n$ such that $[a_{n+1}, a_i] = \lambda_i$ for all $i \leqslant n$. If a_{n+1} belongs to $\mathrm{Span}(a_1, \dots, a_n)$, then $a_{n+1} = \mu_1 a_1 + \cdots + \mu_n a_n$, and we choose $\sigma(a_{n+1}) = \mu_1 b_1 + \cdots + \mu_n b_n$. If not, by Lemma 5.16 we can choose some b_{n+1} in B satisfying the equations $[x, b_1] = \lambda_1, \dots, [x, b_n] = \lambda_n$ and we put $\sigma(a_{n+1}) = b_{n+1}$. Note that extending σ to an isomorphism of V itself can be done using Witt's Lemma [Asc, p. 81]. □

Corollary 5.18. *The theory of V is* $\aleph_0$-categorical, i.e. *it has only one model of countable cardinality, up to isomorphism.*

Proof. The same proof produces a back and forth between any two countable structures having the same theory as V. □

Lemma 5.19. *The theory of V is supersimple of SU-rank* 1.

Proof. By Lemma 5.17, any formula $\phi(x_1, \dots, x_\ell, a_1, \dots, a_m, \lambda_1, \dots, \lambda_n)$ in ℓ variables is a boolean combination of formulas of the form $x_i = a_j$, $x_i = x_j$, $[x_i, x_j] = \lambda_k$ or $[x, a_i] = \lambda_j$. Note that $[x, a] \neq \lambda_j$ is equivalent to the finite disjunction $\bigvee_{\lambda \in F \setminus \{\lambda_i\}} [x, a] = \lambda$. If ϕ is a k-dividing formula over A for some $k \geqslant 2$, then either it is algebraic, or it is implied by a formula of the form $[x, a] = \lambda$. In the second case, $[x, a] = \lambda$ is k-dividing over A. Then there is an A-indiscernible sequence $a_1, a_2 \dots$ such that the conjunction $\bigwedge_{i \in \{1, \dots, k\}} [x, a_i] = \lambda$ is inconsistent. By Lemma 5.16, this means that $a_1, \dots, a_k$ are linearly dependent over F. By indiscernibilty of $a_1, a_2 \dots$, as F is finite, the type of a is algebraic over A, a contradiction. Hence the only forking formulas are algebraic ones. It follows that a non-algebraic ℓ-type can only fork once. □

Lemma 5.20. *Let V be an infinite vector space over the finite field $\mathbf{F}_p$ with p elements, equipped with a skew-symmetric bilinear non-degenerate form. Then V interprets an infinite extra-special p-group.*

Proof. One defines G as follows. G is the set of all pairs $V \times \mathbf{F}_p$ with product

$$(u, a) * (v, b) = \big(u + v, a + b + [u, v]\big).$$

There is no difficulty to check that G satisfies the axioms of Definition 5.15. □

Note that reciprocally, an infinite extra-special p-group interprets an infinite vector space over $\mathbf{F}_p$ equipped with a skew-symmetric bilinear non-degenerate form.

Corollary 5.21. *An infinite extra-special p-group is supersimple of SU-rank* 1 *(and $\aleph_0$-categorical).*

$\aleph_0$-categoricity of infinite extra-special p-groups was first established in [Fel].

The example of extra-special p-groups shows that in Proposition 4.5 and Corollary 4.6, the bound $2n$ cannot be lowered to n. Neither can the bound 2ℓ be replaced by ℓ in Corollary 4.13. More precisely:

Corollary 5.22. *Let G be an infinite extra-special p-group. The theory of G is simple. G has an infinite abelian subgroup. If A is an infinite abelian subgroup of G, then there is no abelian definable subgroup of G containing A.*

Proof. We first claim that no subgroup of finite index in G is abelian. If not, then G has an abelian normal subgroup H having finite index. Let $g_1, \dots g_p$ be a transversal for H. Because G is FC, the group $C_H(g_1) \cap \cdots \cap C_H(g_p)$ has a finite index in H, hence is infinite. But it is included in the centre of G, a contradiction.

To build an infinite abelian subgroup of G, we pick a element g_1 of $G \setminus Z(G)$. As G is FC, any centraliser $C_G(g)$ has a finite index, hence is not abelian by the previous claim. We inductively build countably many pairwise distinct elements $g_1, g_2, \dots$ such that for all $i \geqslant 1$, the element g_{i+1} belongs to $C_G(g_i) \setminus Z\big(C_G(g_i)\big)$. The group $\langle g_1, g_2, \dots \rangle$ is abelian.

The last point is shown in [Plo] using group theoretic methods. We may also use the supersimplicity of G: let A be an infinite abelian subgroup of G, and assume that H is a definable abelian subgroup containing A. The SU-rank of H equals 1, so G/H has SU-rank zero. It follows that H has a finite index in G, but can not be abelian by the first claim. □

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